Another Riemann-Farey Computation

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The Riemann hypothesis is true if and only if

$$R(m) = \sum_{i=2}^{T_m} \left(F_m(i) - \frac{i}{n} \right)^2 = O(m^{-1+\epsilon}) \tag{1}$$

where $F_m(i)$ is the i^{th} element in the Farey sequence of order m and

$$T_m = \sum_{k=2}^{m} \phi(k).$$

Let $P_m(k)$ be sum of the $\phi(k)$ terms in (1) with Farey denominator k so that

$$R(m) = \sum_{k=2}^{m} P_m(k) \tag{2}$$

Figure 1 is a plot of $P_m(k)$ for $m = 500$.

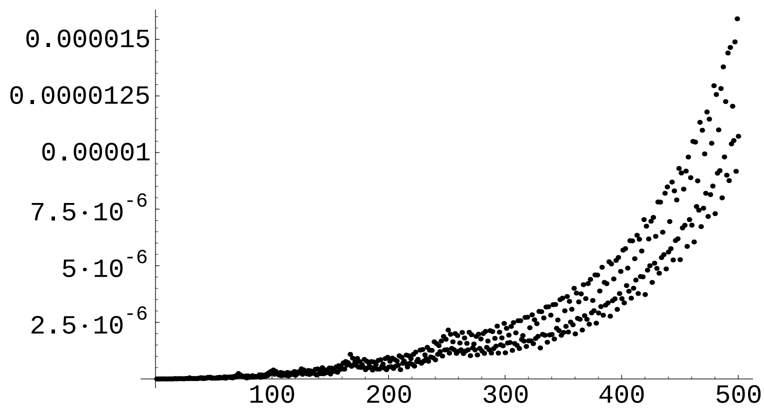

Figure 1: $P_m(k)$ for $m = 500$

Let $Q_m(k, i)$ denote the term with numerator i in $P_m(k)$ so that

$$P_m(k) = \sum_{i=1}^{m} Q_m(k, i) \tag{3}$$

and we note in passing that for all but small values of m and k,

$$Q_m(k, 1) \gg Q_m(k, j) \tag{4}$$

for $j > 1$.

The solid line in Figure 2 is four times $Q_{500}(k,1)$ plotted with $P_{500}(k)$.

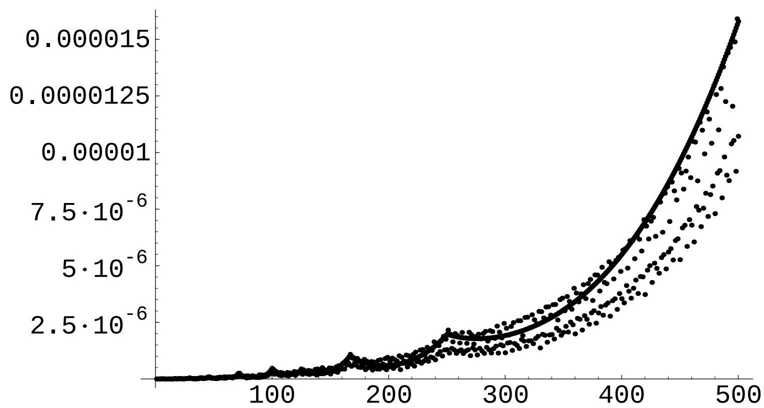

Figure 2: $P_{500}(k)$ and $4Q_{500}(k,1)$

Thus, we are led to consider

$$R(m) < C\sum_{k=2}^{m} Q_m(k,1) \tag{5}$$

For $83 \leq k \leq 500$, $Q_{500}(k,1)$ is given by

$$Q_{500}(k,1) = \begin{cases} \left(\frac{501-k}{76116} - \frac{1}{k}\right)^2, & 250 \leq k \leq 500 \\ \left(\frac{249+2(251-k)}{76116} - \frac{1}{k}\right)^2, & 167 \leq k < 250 \\ \left(\frac{413+4(168-k)}{76116} - \frac{1}{k}\right)^2, & 125 \leq k < 167 \\ \left(\frac{579+6(126-k)}{76116} - \frac{1}{k}\right)^2, & 100 \leq k < 125 \\ \left(\frac{725+10(101-k)}{76116} - \frac{1}{k}\right)^2, & 83 \leq k < 100 \end{cases} \tag{6}$$

Each case in (5) corresponds to a constant step size between adjacent Farey numbers of the form $\frac{1}{k}$. This step size is the factor of $(a-k)$ in the case.

Considering the top case in equation (4), for $\frac{m}{2} \leq k \leq m$ we have in general

$$Q_m(k,1) = \left(\frac{m+1-k}{\sum_{j=2}^{m} \phi(j)} - \frac{1}{k} \right)^2 \tag{7}$$

Substituting the approximation $\frac{3m^2}{\pi^2}$ for the totient sum and integrating from $m/2$ to m, we have

$$\begin{aligned} \hat{R}(m) = &\frac{12\pi^4 + 6m(\pi^4 - 24\pi^2 \log 2) + m^2(216 + \pi^4 - 72\pi^2(\log 4 - 1))}{216m^3} \\ &\approx \frac{0.180m^2 - 1.855m + 5.41}{m^3} \end{aligned} \tag{8}$$

with

$$\lim_{m \to \infty} \hat{R}(m)/m^{-1+\epsilon} = 0 \tag{9}$$

In general, each of the n terms in (1) can be represented as

$$\left(\frac{a+bk}{\sum_{j=2}^{m}\phi(j)}-\frac{i}{k}\right)^2 \tag{10}$$

for $[m/c] \leq k < [m/c]+1$ and a, b and c depending on m, k and i.

As above, substituting the approximation $\frac{3m^2}{\pi^2}$ for the totient sum, integrating from $\frac{m}{c}$ to $\frac{m}{c}+1$ for each term and then summing over the n terms, we have

$$\begin{aligned} R(m) < \hat{R}(m) = C\sum_{k,i} & \frac{27c^4i^2m^3 - 18bc^2im^2(c+m)\pi^2 + 18ac^2im^2(c+m)\pi^2\log\frac{m}{(m+c)}}{27c^2(m+c)m^4} \\ & + \frac{(m+c)(3a^2c^2+3abc(c+2m)+b^2(c^2+3cm+3m^2))\pi^4}{27c^2(m+c)m^4} \end{aligned} \tag{11}$$

where the constant C accounts for the totient sum approximation and

$$\lim_{m\to\infty}\hat{R}(m)/m^{-1+\epsilon} = 0 \tag{12}$$